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Three-Dimensional Containment of Charged Particles by Orthogonal Standing Waves

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It is shown how genuine three-dimensional containment of a single charged particle can be achieved even in the presence of a repulsive uniform space-charge distribution using three mutually orthogonal standing waves or equivalently a resonant cavity. Possible extensions of this technique to plasma containment are also discussed since it would appear that the high energies (on the order of 100 keV) achievable with this system make it worthy of further investigation.

I. INTRODUCTION

As far as is known, genuine three-dimensional containment, on the basis of a single-particle model has not been demonstrated for a high voltage, tuned cavity device which employs both E and B fields for containment and exploits the Q of the cavity for the purpose of establishing large fields; although an rf mechanism has been proposed as a possible explanation of naturally occurring ball lightning.[1] However, many two-dimensional arrangements have been considered, most of them relying upon cylindrical geometry.[2] In the majority of these cases the θ motion is superfluous, the r motion is stable, and the z motion ignored. In addition, several schemes have been put forward utilizing radiation pressure, and would obviously have application to very dense, highly conducting plasma configurations. However, practical questions relating to the initial plasma injection at the required densities are usually neglected, along with problems such as the detuning of the cavity as a result of local plasma instabilities, etc. Low voltage, low power, and pure oscillating electric field devices, such as those of Wuerker *et al.*[3] and Shapiro and Watson[4] are generally used for the containment of particles possessing an e/m ratio of less than unity and, therefore, will not be discussed.

Since the entire problem of genuine plasma containment is exceedingly complex, an enormous simplification is afforded by starting the analysis from the single-particle equations. This allows us to be unconcerned about the many-body aspects of plasma physics, such as charge separation, polarization, and conductivity. Later, however, we shall include space-charge effects due to a uniform distribution of particles of the same sign, and finally attempt to make some qualitative remarks about the expected behavior of an actual plasma system.

It is the purpose of this paper to show how three mutually orthogonal standing waves can give rise to genuine containment of a single charged particle even in the presence of a repulsive uniform space-charge distribution. One normal mode of the system is demonstrated to have a Mathieu-type solution and has regions of stability depending upon the choice of various parameters. Since it was impossible to obtain an analytic solution of the whole problem, we used analog computer solutions to demonstrate the nature of the solutions as a function of the various parameters at our disposal, such as frequency and field strength.

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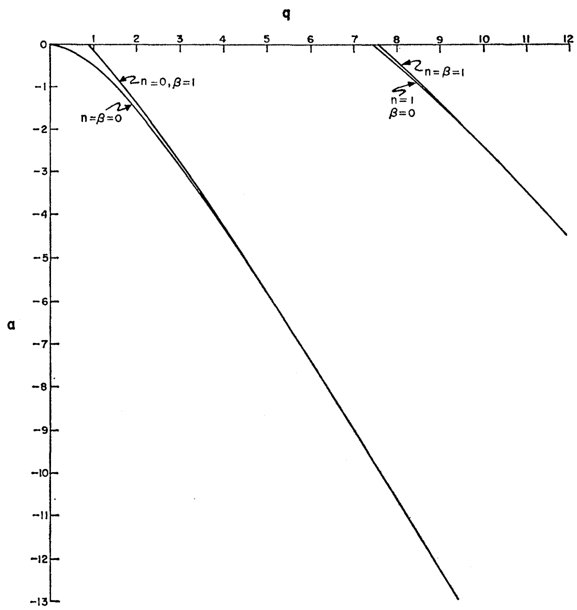

FIG. 1. Stability regions in the $a-q$ plane for a charged particle in the neighborhood of a cloud of like particles.

II. DEVELOPMENT

A. Single-Particle Equations

Without actually describing their physical origin, we shall consider three mutually orthogonal standing-waves represented by the following fields:

$$B_{x_i}=-B_0 \sin\omega t \cos\frac{2\pi x_{(i+2)}}{\lambda}, \qquad (1)$$

$$E_{x_i}=cB_0 \cos\omega t \sin\frac{2\pi x_{(i+1)}}{\lambda}, \qquad (2)$$

plus cyclic permutations, $i=1, 2, 3$, where B_0 is the maximum value of the oscillating magnetic fields of angular frequency ω and wavelength λ. Notice that pairs of these equations independently satisfy Maxwell's equations and that the E and B fields are $\pi/2$ out of phase in both space and time, characteristic of the standing-wave form.

Using the familiar Lorentz force equation

$$F_{x_i}=e(E_{x_i}+g_{ijk}\dot{x}_j B_{x_k}), \qquad (3)$$

and concerning ourselves with small displacements in the vicinity of an electric node, we may readily derive the following equations of motion for a charged particle of mass m and charge e:

$$\ddot{x}_i=(k/\omega)(\dot{x}_{i+1}-\dot{x}_{i+2}) \sin\omega t+kx_{i+1} \cos\omega t, \qquad (4)$$

where we have set $eB\omega/m=k$ and have replaced $\sin(2\pi x/\lambda)$ by $2\pi x/\lambda$, and $\cos(2\pi y/\lambda)$ by unity. We notice immediately upon addition that the quantity

$$\xi=\sum_{i=1}^{3} x_i \qquad (5)$$

satisfies the normalized Mathieu equation

$$\xi''+(a-2q\cos 2\tau)\xi=0, \quad a=0, \quad q=2k/\omega^2, \quad \tau=\omega t/2. \qquad (6)$$

The regions of stability are characterized by the usual Mathieu plot (see Fig. 1). In this case, $a=0$ and stable solutions exist in the first region for $k<0.45\omega^2$. However, this is not necessarily sufficient to guarantee the boundedness of the separate coordinates individually.

It will suffice to say that the other normal modes are

$$\eta=x_1+e^{i\pi/3}x_2+e^{-i\pi/3}x_3, \qquad (7)$$

$$\zeta=x_1+e^{-i\pi/3}x_2+e^{i\pi/3}x_3=\eta^*, \qquad (8)$$

and satisfy Hill equations with complex coefficients. Attempts to solve the resulting equations for $x_i(t)$, $\dot{x}_i(t)$ analytically have failed. However, particular solutions have been obtained by means of an analog computer and are presented in Sec. C.

B. Space Charge

The effects of a uniform space charge upon the equations of motion of a single particle can be demonstrated by considering a spherical charge distribution of the same sign and of uniform density N particles/m^3 located at $x_i=0$.

A simple consequence of Gauss' theorem indicates that the force experienced by a test particle at position x_i inside the charge distribution is due to those charges between x_i and the origin, so that

$$F_{x_i}=\frac{Ne^2}{4\pi\epsilon_0 x_i^2}(\tfrac{4}{3}\pi x_i^3)=\frac{Ne^2}{3\epsilon_0}x_i. \qquad (9)$$

These forces are all repulsive and modify Eqs. (4) to a form given by

$$\ddot{x}_i=kx_{i+1}\cos\omega t+\frac{k}{\omega}(\dot{x}_{i+1}-\dot{x}_{i+2})\sin\omega t+\frac{Ne^2}{3m\epsilon_0}x_i. \qquad (10)$$

As an indication, and for the sake of simplicity, addition of the above three sets of equations yields, in normalized form,

$$\xi''+(a-2q\cos 2\tau)\xi=0, \qquad (11)$$

where

$$a=\frac{-4Ne^2}{3m\omega^2\epsilon_0}, \qquad (12)$$

with q, τ, and ξ defined as in Eqs. (5) and (6).

Thus, the space charge may be expected to provide a negative "a" term which limits operation to those portions of the stability regions lying below the q axis (narrow regions between converging lines in Fig. 1). For example, using Fig. 1, we find that a representative choice of parameters for stability is $a=-0.2$, $q=1$. These parameter values can be achieved at a drive frequency of 10^{10} rad/sec, with a field strength of 6.2×10^{6} V/m and an electron density of about $5\times10^{15}/\mathrm{m}^3$.

From a purely qualitative point of view, it is clear that the introduction of positive ions into the system has a twofold effect: (1) It tends to restore the effective a of Eq. (11) back to zero, and (2) gives rise to a polarization phenomenon which tends to reduce the effective fields inside the plasma. In short, it restores a to zero, but reduces the effective value of q, in Eq. (11). In fact, the reduction in the field strength could be computed, given the effective dielectric constant. It

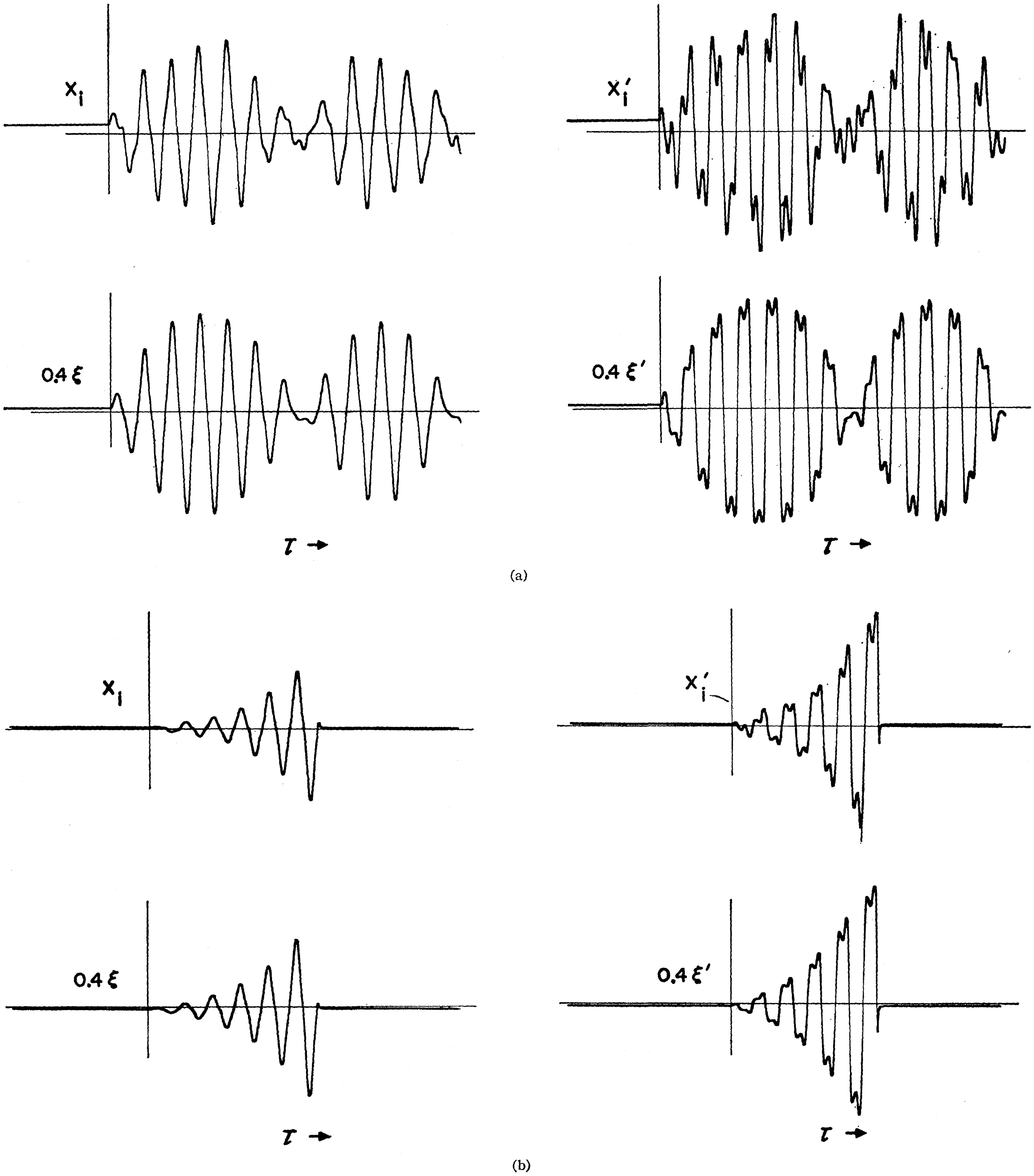

FIG. 2. Computer solutions of Eqs. (4) for $n=0$ and the initial conditions: $x_i=1,\ 2,\ -2.5$; $x_i'=2,\ -1.5,\ 2.5$. (a) $a=-0.115$, $q=1$; (b) $a=-0.11$, $q=1$.

should perhaps be stressed that the electron cloud picture represents the worst possible plasma configuration, and that in a physically realizable plasma, one would expect densities at least an order of magnitude greater than those derived on the basis of the pure electron cloud distribution model.

C. Solutions

Equations (4) were solved on an analog computer (GEDA). These equations are linear and hence the form of the solutions is independent of initial velocity and displacement (see Sec. D). However, one must bear in mind that Eqs. (4) correspond to the physical situation only for

$$0<x_i/\lambda<0.1.$$

The primary purpose of the computer simulation was to determine whether the solutions to Eqs. (4) have regions of stability similar to those for Mathieu equations.

The typical computer solutions shown in Fig. 2 indicate that within the accuracy of the computer the solutions in the x_i and $\dot{x}_i$ are identical to those in ξ and $\dot{\xi}$. Furthermore, all the solutions exhibit those characteristics peculiar to Mathieu functions. The solutions shown in Fig. 2 were obtained with different initial conditions in each coordinate x_i. For equal initial conditions the solutions for x_i bear an even greater resemblance to those for the ξ normal mode.

The boundaries of the first stability region for the Mathieu equation were explored for Eqs. (4), i.e., $0\leq\nu\leq1$. It was found that they behaved as if they were Mathieu equations. Figure 2(b) illustrates the onset of instability at $a=-0.11$, $q=1$.

In Sec. D, it will be shown that the achievable kinetic energy is a function of the parameter $(n+\beta)$. Figure 2(a) illustrates the solution for a combination of field strength and Coulomb repulsion for which $n=0$, $\beta\sim1$. Thus, stability is indicated for these high-energy configurations.

D. Kinetic Energy

We have shown that the equation of motion of a single charged particle at the intersection of three mutually orthogonal standing electromagnetic waves may be represented by the Mathieu equation, Eq. (6). The general stable solution of this equation may be written

$$x_i=A_i\sum_{r=-\infty}^{\infty}D_{2r+p}\cos(2r+p+\beta)\tau+B_i\sum_{r=-\infty}^{\infty}D_{2r+p}\sin(2r+p+\beta)\tau, \quad (13)$$

$$A_i=x_i(0)/\sum_{r=-\infty}^{\infty}D_{2r+p}, \quad (14)$$

$$B_i=x_i'(0)/\sum_{r=-\infty}^{\infty}(2r+p+\beta)D_{2r+p},\quad x_i'(0)=2\dot{x}_i(0)/\omega \quad (15)$$

$$0\leq\beta\leq1,$$

where p is 0 or 1, depending upon the region in the $a-q$ plane and β is the fractional portion of the nondimensional characteristic $\nu=n+\beta$, n an integer.[5] The A_i and B_i are determined by the initial coordinates, $x_i(0)$, $\dot{x}_i(0)$, and the series are Mathieu functions of order ν.

The kinetic energy, T, of this particle is

$$T=\sum_{i=1}^{3}T_i$$

$$T_i=\frac{\omega^2}{8}m(A_i^2+B_i^2)\left\{\sum_{r=-\infty}^{\infty}D_{2r+p}(2r+p+\beta)\times\sin\left[(2r+p+\beta)\tau-\tan^{-1}\left(\frac{B_i}{A_i}\right)\right]\right\}^2.$$

Assuming, for simplicity, equal initial conditions in the x_i and substituting from Eqs. (14) and (15), we obtain

$$T=\frac{3\omega^2}{8}m\left\{x(0)^2\{\sum_{r=-\infty}^{\infty}D_{2r+p}[(2r+p+\beta)\sin(2r+p+\beta)\tau-\tan^{-1}\psi]/\sum_{r=-\infty}^{\infty}D_{2r+p}\}^2+\frac{4\dot{x}(0)^2}{\omega^2}\{\sum_{r=-\infty}^{\infty}D_{2r+p}(2r+p+\beta)\sin[(2r+p+\beta)\tau-\tan^{-1}\psi]/\sum_{r=-\infty}^{\infty}(2r+p+\beta)D_{2r+p}\}^2\right\}$$

$$\psi=\frac{2\dot{x}(0)}{\omega x(0)}\sum_{r=-\infty}^{\infty}D_{2r+p}/\sum_{r=-\infty}^{\infty}(2r+p+\beta)D_{2r+p}.$$

An estimate of the energy may be obtained from the following considerations:

(a) The products of terms at different frequencies contribute only to the instantaneous energy; their integral over a period vanishes, and

$$\sin^2 n\tau>\sin n\tau\,\sin l\tau,\quad n\neq l.$$

[5] N. W. McLachlan, *Theory and Application of Mathieu Functions* (Oxford University Press, New York, 1947), p. 77.

(b) For $q<5$, the coefficient of the term for which $2r+p=n$ is numerically greater than the others,[6] i.e.,

$$|D_n|>|D_{2r+p}|,\quad -\infty<r<\infty,\quad q<5.$$

(c) Equations (4) are valid only in the range

$$0\leq\alpha\leq 0.1,\quad \alpha_i=x_i/\lambda$$

and from physical reasoning the particle may be contained for a value of α up to 0.25. Therefore,

$$T\approx\frac{3\omega^2}{8}m\left[x(0)^2(n+\beta)^2+\frac{4}{\omega^2}\dot{x}(0)^2\right].$$

Let

$$x(0)=\alpha\lambda$$

and recalling that

$$\omega=2\pi c/\lambda$$

we obtain the following approximate expression for the kinetic energy of the system

$$T\approx\frac{3\pi^2}{2}mc^2\left[\alpha^2(n+\beta)^2+\frac{\dot{x}(0)^2}{\pi^2c^2}\right]. \tag{16}$$

Equation (16) clearly shows that the mean kinetic energy is only weakly dependent upon the initial injection velocity, and is almost entirely governed by the initial displacement and the stability parameters. Therefore, a reasonable estimate of the kinetic energy of a single electron contained in the first stability region of the Mathieu equation, i.e., $n=0$, will lie between 75 and 474 keV, corresponding to α of 0.1 and 0.25, respectively.

Consideration of the space-charge effect as in the previous section will restrict operation to $a\leq 0$. Whereas this does restrict the choice of n, it does not affect the available range of β. For $n=0$, there is an appreciable region of stability below the q axis (see Fig. 1). One could conceivably operate at higher values of n as shown in Fig. 1. Moreover, it is interesting to note that despite the critical requirements on a and q for stability in the region $n=1$, $a\leq 0$, the stable range of kinetic energies is 4 to 1, i.e., $(n+1)^2/(n+0)^2$.

III. RESONANT CAVITY

From a purely physical point of view, it would be desirable to show that the equations of motion, Eqs. (4), may be obtained from the use of particular modes of a resonant cavity. This may be accomplished by investigating the properties of the following field configurations:

$$E_{x_i}=E_0\sin\frac{2\pi x_{(i+1)}}{d}\sin\frac{\pi x_{(i+2)}}{d}\cos\omega t. \tag{17}$$

[6] National Bureau of Standards, *Tables Relating to Mathieu Functions* (Columbia University Press, New York, 1951).

We notice that $\nabla\cdot\mathbf{E}=0$ is satisfied identically, moreover, we can find the associated B fields from the solution

$$\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t \tag{18}$$

giving rise to

$$B_{x_i}=\frac{E_0\pi}{\omega d}\cos\frac{\pi x_{i+1}}{d}\sin\frac{2\pi x_i}{d}-2\sin\frac{\pi x_i}{d}\cos\frac{2\pi x_{i+2}}{d}\sin\omega t. \tag{19}$$

One readily observes that

$$\partial_i B_{x_i}=0. \tag{20}$$

In order to put these waves into a cubical cavity of side d, we require, furthermore, that

$$\nabla\times\mathbf{H}=\mu_0\epsilon_0(\partial\mathbf{E}/\partial t). \tag{21}$$

Simple substitution of the appropriate quantities gives rise to the condition

$$\lambda_g=[2/(5)^{1/2}]d. \tag{22}$$

We now find it convenient to displace our origin of coordinates from one corner of our cavity to the center of the cube, necessitating the replacement of x_i by $x_i'-\frac{1}{2}d$. Thus, we obtain, finally,

$$E_{x_i}=-E_0\cos\frac{\pi x_{i+2}'}{d}\sin\frac{2\pi x_{i+1}'}{d}\cos\omega t. \tag{23}$$

Similarly, $B_{x_i'}$ is given by

$$B_{x_i'}=-\frac{E_0\pi}{d\omega}\sin\frac{\pi x_{i+1}'}{d}\sin\frac{2\pi x_i'}{d}+2\cos\frac{\pi x_i'}{d}\cos\frac{2\pi x_{i+2}'}{d}\sin\omega t \tag{24}$$

plus appropriate permutations.

Before deriving the equations of motion, we shall again assume small displacement (i.e., $2\pi x_i/\lambda\ll 1$, etc.) and ignore quadratic terms in the displacement; i.e., Eqs. (23) and (24) may be approximated by

$$E_{x_i}\simeq -E_0\frac{2\pi x_{i+1}}{d}\cos\omega t \tag{25}$$

and

$$B_{x_i}\simeq -E_0\frac{2\pi}{d\omega}\sin\omega t. \tag{26}$$

Using the Lorentz force equation (3), we readily obtain the equations

$$\ddot{x}_i=-kx_{i+1}\cos\omega t-\frac{k}{\omega}(\dot{x}_{i+1}-\dot{x}_{i+2})\sin\omega t, \tag{27}$$

where we have dropped the primes for convenience. We note that the above set is identical to Eqs. (4), except for the replacement of k by $-k$, corresponding to changing the sign of the charge. This, of course, has no effect on the boundedness of the solutions since k is multiplied by a periodic function. Hence, by analogy with the original set of equations of motion, we have containment of the charged particle with the same range of parameters, provided we recall that we are assuming small displacements from the center of the cube.

The input power P, required to achieve a given operating point in the $a-q$ plane is a function of the Q of the cavity. The Q may be expressed in terms of the conductivity of the walls, σ, as

$$Q=2(\pi c\mu_0\lambda\sigma)^{1/2}. \tag{28}$$

The amplitude of the electric field in each of the three modes that must be excited in the cavity is obtained from

$$E_i^2=\frac{P_iQ}{2\pi\epsilon_0 c\lambda^2}. \tag{29}$$

Substituting in Eqs. (4) and (6), we obtain the Mathieu parameter

$$q_i=\frac{e}{m}\left(\frac{2P_i}{\epsilon_0}\right)^{1/2}\left(\frac{\mu_0\lambda\sigma}{\pi^3c^9}\right)^{1/4} \tag{30}$$

in terms of the power input, the wavelength, and the wall conductivity. At $\omega=10^{10}$ rad/sec, a q of 0.342 in each of the three coordinates would be achieved with an input power of 300 kW and a wall conductivity of 3×10^6 mho/m. The σ chosen corresponds to a cavity Q of 50 000 at this frequency.

IV. CONCLUSION

We have shown, using analog computer techniques, that a single charged particle can be contained inside a resonant cavity, provided that (1) certain boundary conditions are employed and (2) that the system is operated in an appropriate region of stability for the normal mode that is described by a Mathieu equation.

It is estimated that a single electron could be contained in the neighborhood of a central pure electron cloud of density 10^{16} electrons/m^3 by an rf source supplying 2.5 MW at 1.6 Gc/sec and a cavity with a Q of 50 000. The resulting normal mode parameters would be $q=1$, $a=-0.4$, and the electron would have an energy on the order of 50 keV. The energy of this particle and the density of the cloud are linear with input power. However, the wavelength dependence of the energy is $\lambda^{1/2}$ and that of the cloud density $\lambda^{3/2}$. Moreover, the Mathieu stability parameter q is proportional to the square root of the input power and to the fourth root of the containing wavelength.

It should perhaps be mentioned that the equations of motion have been derived on the basis of neglecting the radiation reaction,[7] relativistic effects and the well-known diffusion mechanisms. The former effect will produce a drag on the electron and give rise to a nonzero expression for the power dissipated through the particle. The relativistic effects will be expected to remain insignificant ($<10\%$) provided that we concern ourselves with fields less than 150 kV, which will certainly be the case for electrons trapped in the first region of stability. The inclusion of diffusion effects is presently being considered.

Finally, collective plasma oscillations will tend to become important when the plasma frequency ω_p is comparable or larger than the drive frequency ω. The replacement of the charge ρ and current ρv by the usual velocity integrals of distribution functions $F_{i,j}(x,v)$ of electrons and ions would lead to a Vlasov-type equation and would obviously have to be solved digitally. Experience indicates[8] that this is probably too lengthy for existing machines, hence an experimental program would probably be more fruitful. From a practical point of view, it is clear that it is necessary to increase the Q of the cavity (e.g., by supercooling) in order to minimize the prohibitive power requirements.

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[7] Giving rise to the familiar runaway solutions, but which can be eliminated by judicious choice of the boundary conditions, e.g., see R. E. Norton and W. K. R. Watson, Phys. Rev. **116**, 1597 (1959).

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